Optimal Recovery of Elastic Properties for a General Anisotropic Material through Ultrasonic Measurements

Miao Sun
Michael "Mick" Peterson

Mechanical Engineering
University of Maine

Motivation

- Full knowledge of elastic and visco-elastic properties of the material are critical in many design applications.
- Dynamic response and design of materials require full properties of the material.
- Warpage in manufacturing thermal expansion coupled loading.

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Outline of the Thesis (I)

- Measure the phase velocity of an ultrasonic wave propagated through the testing material using immersion technique.
- Reconstruct 21 elastic constants from measured phase velocities. (*Numerical Optimization).*
- On structure of the elastic symmetry of an anisotropic material.
  - *Determine normals to the symmetry planes.*
  - *Recovery of principal coordinate system. Find Euler's angles*

Outline of the Thesis (II)

- Signal processing.
  - *Relative time delay estimation-Cross correlation.*
- Experimental Setup.
- Results and discussion.
  - *Results of Reconstructed elastic constants.*
  - *Results of misorientation between geometric axes and symmetric axes.*
- Conclusion and Future work.

Immersion Technique

- What is immersion technique?
  - *Measure phase velocity of an obliquely incident ultrasonic wave propagated from liquid into sample.*
- Immersion technique has more advantage over the contact technique.
  - *The sample does not have to be cut in different directions.*
  - *The coupling fluid (water) is well behaved at all incidental angles.*
- Resulting data is velocity verse incident angle.

Waves in Anisotropic Solid

- Constitutive Equation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \xrightarrow{\text{substituted into}} \left.\begin{array}{l} \sigma_{ij,j} = \rho \ddot{u}_i \\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \end{array}\right\} C_{ijkl}\, u_{k,jl} = \rho\, \ddot{u}_i$$

C_{ijkl} has at most 21 independent components.

- Assuming a solution of the form

$$u_k = A_0\, p_k e^{i(k_r x_r - \omega t)}$$

- Christoffel's equation

$$\left[\Gamma_{ik} - \delta_{ik}\rho V^2\right]\{p_k\} = 0 \qquad \Gamma_{ik} = C_{ijkl} n_j n_l : \textit{Christoffel's tensor}$$

$$n_l, n_j : \textit{direction cosines}$$

Solution of Christoffel's Equation

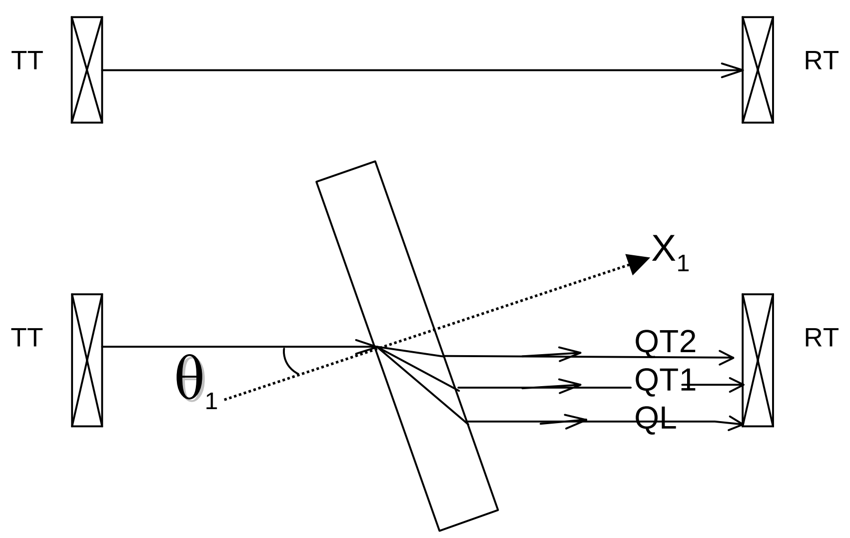
$$\left[\Gamma_{ik} - \delta_{ik}\rho V^2\right]\{p_k\} = 0 \Rightarrow \left|\Gamma_{ik} - \delta_{ik}\rho V^2\right| = 0$$

- The phase velocities are the three eigenvalues (solutions) of Christoffel's equation.
- Three modes can be excited along different velocities and polarizations.

--one quasi-longitudinal (QL), a fast (QT1) and a slow (QT2) quasi-transverse wave

Wave Velocity Measurement by Immersion Technique (I)

Measurements are performed by ultrasonic pulses through reference media (water) and an anisotropic sample at a non-normal incident angle.



Wave Velocity Measurement by Immersion Technique (II)

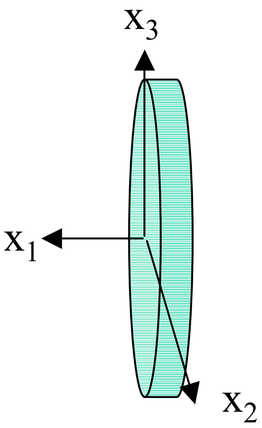
Experiment data are collected from 4 incident planes associated with different incident angle.

What is an incident plane?

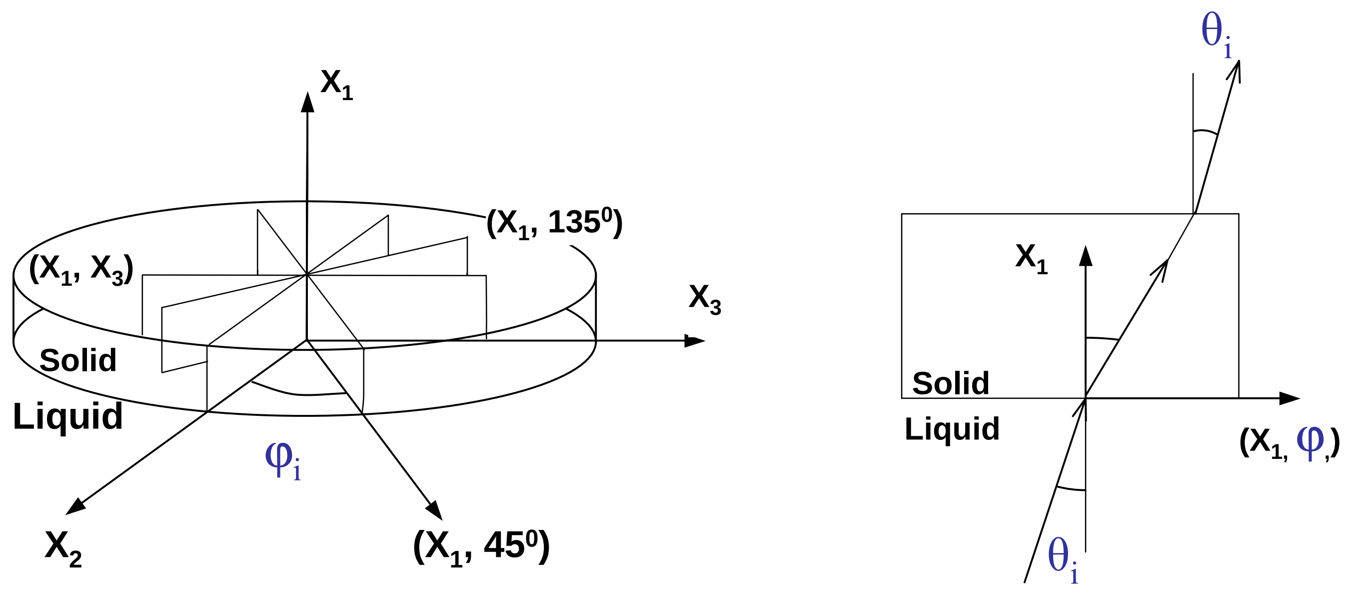
- *A plane consists of the* $\mathbf{x_1}$ *and incident wave.*

What is an incident angle?

- *An angle between the* $\mathbf{x}_1$ *and incident wave.*



Incident Plane (x_1, φ_i) and Incident Angle (θ_i)



Incident planes are defined by (x_1, φ_i)

$\varphi_i = 0^0, 45^0, 90^0, 135^0$: azimuthal angle

Reconstruction of C_{ij} From the Wavespeed Data (Inverse Problem)

$$\left[\Gamma_{ik} - \delta_{ik}\rho V_i^{\,2}\right]\{p_k\} = 0 \Rightarrow \left|\Gamma_{ik} - \delta_{ik}\rho V_i^{\,2}\right| \cong 0$$

$$\Gamma_{ik} = C_{ijkl} n_j n_l \; : Christoffel's\ Tensor$$

- Non-linear cubic equation in $\rho V_i^{\,2}$
- Inverse problem of constructing a function to minimization
- Optimization problem
  - *Newton-Raphson method*

Newton-Raphson Approach

$$f(V_{\exp}, C_{cal}) = \left| \Gamma_{ij} - \delta_{ik} \rho V^2 \right| \cong 0 \qquad \Gamma_{ik} = C_{ijkl} n_j n_l$$

- Define an objective function F ($V_{exp,}$ C_{cal})--*sum of square of* $f(V_{exp},C_{cal})$.

$$F(V_{\exp}, C_{cal}) = \sum_{i=1}^{n} \left[f(V_{\exp}(i), C_{cal}) \right]^2$$

- C_{ij} → Minimizing objective function.

$$C_{ij} = \min \left\{ F(V_{\exp}, C_{cal}) \right\}$$

C_{cal} → Calculated data

V_{exp} → Experimental data

- Procedure may be iterated until

$$e_k - e_{k-1} = C_{cal}^{k} - C_{cal}^{k-1} \le \varepsilon$$

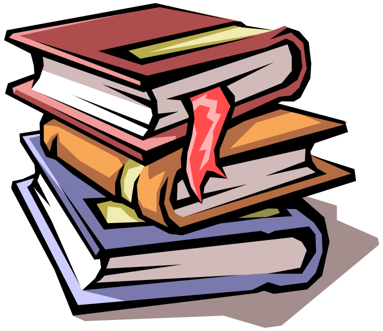
- Requires no mode distinction.
  - *Both shear and longitudinal modes generated.*
  - *Both modes are solutions of Christoffel's equation.*

Elastic Constants Found

- Elastic constants have been found.
  - Using immersion technique, measured phase velocity verse incident angle.
  - Reconstructed elastic constants from experimental data

    Optimization approach.
  - *Elastic constants are in an observation coordinate system R.*
- Next
  - *Recovery symmetry planes*

    as well as the principal axes.



Structure of Elastic Symmetry of an Anisotropic Material (I)

- Two second order tensors A_{ij} (Voigt tensor) and B_{ij} (dilatational modulus) defined from C_{ijkl} are required.

$$A_{ij} = C_{ijkk} \ , B_{ij} = C_{ikjk}$$

$$A = \begin{bmatrix} C_{11} + C_{12} + C_{13} & C_{16} + C_{26} + C_{36} & C_{15} + C_{25} + C_{35} \\ C_{16} + C_{26} + C_{36} & C_{12} + C_{22} + C_{23} & C_{14} + C_{24} + C_{34} \\ C_{15} + C_{25} + C_{35} & C_{14} + C_{24} + C_{34} & C_{13} + C_{23} + C_{33} \end{bmatrix} \quad B = \begin{bmatrix} C_{11} + C_{55} + C_{66} & C_{16} + C_{26} + C_{45} & C_{15} + C_{46} + C_{35} \\ C_{16} + C_{26} + C_{45} & C_{22} + C_{44} + C_{66} & C_{24} + C_{34} + C_{56} \\ C_{15} + C_{46} + C_{35} & C_{24} + C_{34} + C_{56} & C_{33} + C_{44} + C_{55} \end{bmatrix}$$

- A vector is normal to a symmetry plane of the material if and only if the vector is an eigenvector of tensor A and B respectively– [Cowin, 1980].

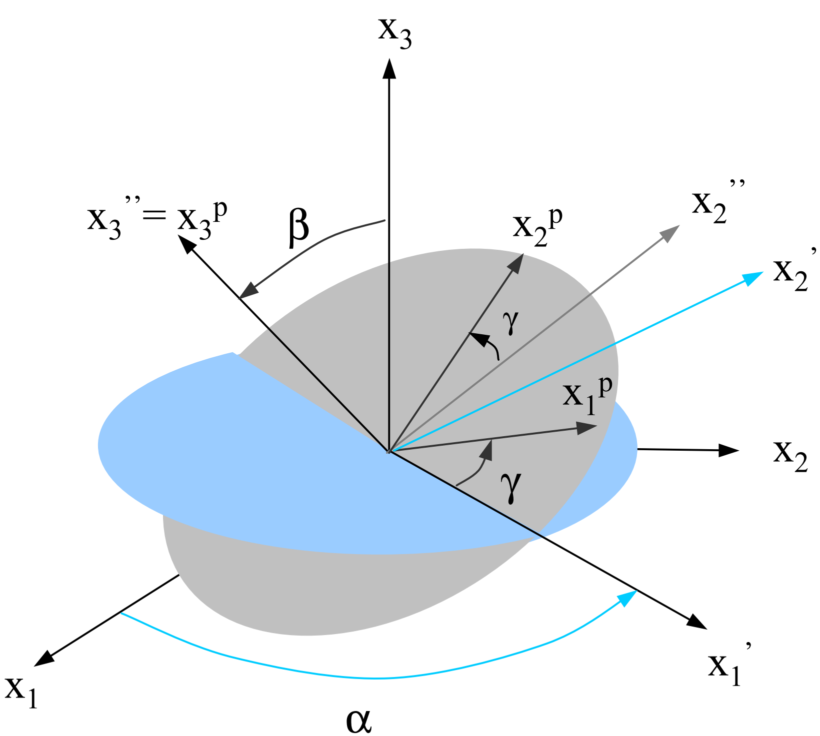
Structure of Elastic Symmetry (II)

- Theoretically, if
  - Three eigenvectors coincide--*Orthotropic.*
  - One eigenvector coincides--*monoclinic.*
  - None coincides --*triclinic.*
- Due to the experimental errors, the eigenvector pairs of A and B do not exactly line up.
  - A normal is contained in an angle around the average directions between the closest eigenvectors of A and B.
  - An eigenvector pair
    - Small angular deviation -- normal to a symmetry plane.
    - What is small deviation?

Recovery of Principal Coordinate System

- Measured elastic constants are in an observation coordinate system R.
  arbitrary choice prior to the measurement.
- Orientation of principal coordinate system R^P with respect to R --
  specified by Euler's angles $\delta = (\alpha, \beta, \gamma)$.
- R^P ⇧ R through at most 3 successive rotations.

Euler's Angles



- R with axes x_1, x_2, x_3 is clockwise rotated through α about the x_3 ⇧R'
- Clockwise rotation through β about x_1' ⇧R''.
- Final clockwise rotation γ about x_3'' ⇧R^P with x_1^p, x_2^p, and x_3^p.

Euler's Angles (II)

$$\{x_i{}^p\} = [a(\alpha)][a(\beta)][a(\gamma)]\{x_i\} = [M]\{x_i\}$$

$$[a(\alpha)] = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[a(\beta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & \sin(\beta) \\ 0 & -\sin(\beta) & \cos(\beta) \end{bmatrix}$$

$$[a(\gamma)] = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Locating R^P equivalent to determining $\delta = (\alpha, \beta, \gamma)$.
- In general, two rotations are enough to characterize the orientation of R^P.
- At least, one of the Euler's angle can be assumed to be zero. If $\gamma=0$, then:

$$[M] = \begin{bmatrix} c\alpha\, c\gamma - s\alpha\, c\beta\, s\gamma & s\alpha\, c\gamma + c\alpha\, c\beta\, s\gamma & s\beta\, s\gamma \\ -c\alpha\, s\gamma - s\alpha\, c\beta\, c\gamma & -s\alpha\, s\gamma + c\alpha\, c\beta\, c\gamma & s\beta\, c\gamma \\ s\alpha\, s\beta & -c\alpha\, s\beta & c\beta \end{bmatrix}$$

$$c = \cos;\ s = \sin$$

$$[M] = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha)\cos(\beta) & \cos(\alpha)\cos(\beta) & \sin(\beta) \\ \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) & \cos(\beta) \end{bmatrix}$$

An Example to Illustrate Determining Euler's Angles

R^P is first rotated 30^0 about $\mathbf{x_3}$, then a rotation of 10^0 about transformed x_1 is performed to bring into a system R''.

$$C = \begin{bmatrix} 47.601 & 44.371 & 46.342 & 0 & 0 & 0 \\ & 18.906 & 20.327 & 0 & 0 & 0 \\ & & 18.839 & 0 & 0 & 0 \\ & & & 3.1112 & 0 & 0 \\ & \text{sym.} & & & 5.2180 & 0 \\ & & & & & 4.5369 \end{bmatrix}$$

$$C'' = \begin{bmatrix} 47.999 & 36.891 & 39.747 & 0.520 & 0.320 & -1.813 \\ & 33.665 & 26.371 & -0.003 & 1.534 & -10.498 \\ & & 19.745 & -2.530 & 2.260 & -11.019 \\ & & & 3.178 & -0.824 & -0.264 \\ & \text{sym.} & & & 4.459 & 1.321 \\ & & & & & -2.802 \end{bmatrix}$$

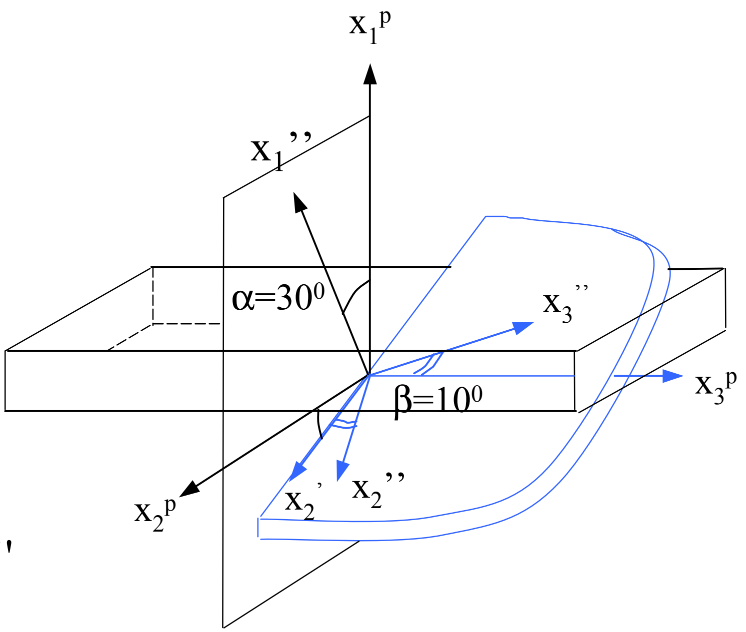
An Example to Illustrate Determining Euler's Angles (II)

Normals to symmetry planes with respect to R'' are

$$e_1{}'' = 0.5\ x_1{}'' + 0.8529\ x_2{}'' - 0.1504\ x_3{}''$$
$$e_2{}'' = 0\ x_1{}'' + 0.1736\ x_2{}'' + 0.9848\ x_3{}''$$
$$e_3{}'' = 0.866\ x_1{}'' - 0.4924\ x_2{}'' + 0.0868\ x_3{}''$$

$\delta = (\alpha, \beta, \gamma)$ can be extracted from $e_3{}''$

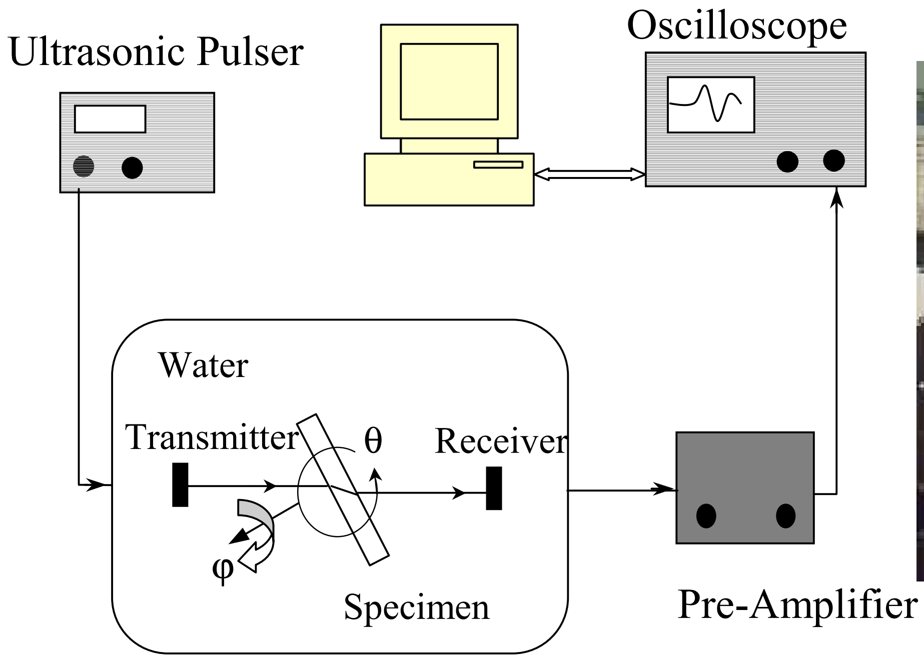
$$e_3{}'' = 0.866\, x_1{}'' - 0.4924\, x_2{}'' + 0.0868 x_3{}''$$
$$= \cos(\alpha)\, x_1{}'' - \sin(\alpha)\cos(\beta)\, x_2{}'' + \sin(\alpha)\sin(\beta)\, x_3{}''$$



Therefore

$$\alpha = 30^0, \beta = 10^0, \gamma = 0$$

Experimental Setup



Transducers and samples are all immersed in water.

Experimental Measurements

- The measurements are performed in 4 incident planes(x_1, φ), where azimuthal angle $\varphi=0^0, 45^0, 90^0, 135^0$.
- In each incident plane, 25 measurements are carried out corresponding to 0^0-24^0 incident angles.
- At each incident angle, relative time delay is measured for signal received.
- A single crystal of aluminum oxide and three carbon-carbon samples are tested in this system.

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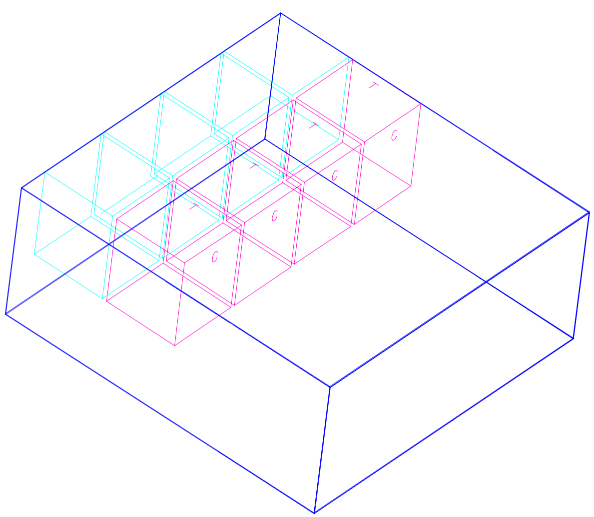
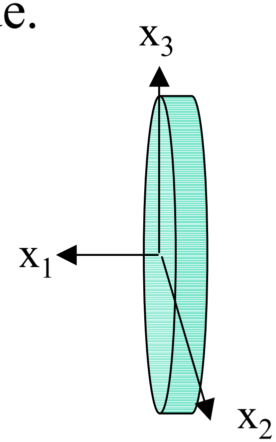
Choice of Materials

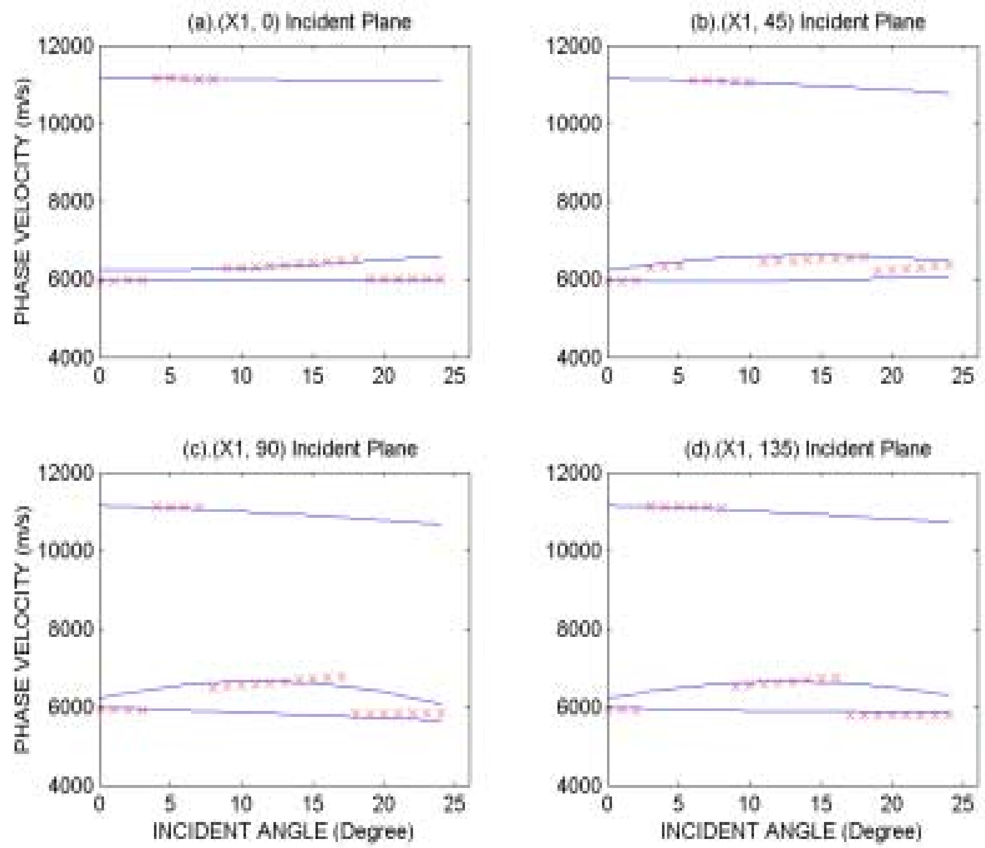


- Aluminum oxide (Φ25.4mm×3mm) has a trigonal system with 6 elastic constants. Handbook values are used as initial guess for the algorithm.
- The carbon-carbon composite (25mm×25mm×6mm) are provided by ATS. The properties are obtained from the initial work using contact technique with assuming they are orthotropic material.

Results of Measured Phase Velocities for Aluminum Oxide

Theoretical phase velocities (in m/s) calculated from the reconstructed elastic constants are compared with the measured velocities for aluminum oxide.





Results of Reconstructed Elastic Constants for aluminum oxide

- Reconstructed elastic (in GPa) constants in the observation coordinate system R is:

$$[C]=\begin{bmatrix} 471.184 & 158.660 & 135.140 & -45.982 & 0.1091 & 0.0010 \\ & 492.222 & 118.221 & 3.444 & 00.4033 & 0.3800 \\ & & 348.080 & 4.033 & 0.6200 & 0.1006 \\ & & & 140.930 & 0.4420 & 0.4801 \\ & sym. & & & 41.410 & -81.340 \\ & & & & & 143.020 \end{bmatrix}$$

- Eigenvectors of tensor A (Voigt tensor) and B (dilatational tensor) are:

$$[Eig\_A]=\begin{bmatrix} -0.0074 & -0.9996 & -0.0275 \\ 0.2137 & 0.0254 & -0.9766 \\ 0.9769 & -0.0128 & 0.2162 \end{bmatrix}$$

$$[Eig\_B]=\begin{bmatrix} -0.0096 & -0.9999 & -0.0040 \\ 0.2674 & 0.0013 & -0.9636 \\ 0.9635 & -0.0101 & 0.2673 \end{bmatrix}$$

Three angular deviations between each pairs of eigenvectors are:

$$\theta_1 = 3.03^0, \theta_2 = 1.40^0, \theta_3 = 3.35^0$$

Each angular deviation may be considered to be small enough to conclude that three symmetry planes exist within the material.

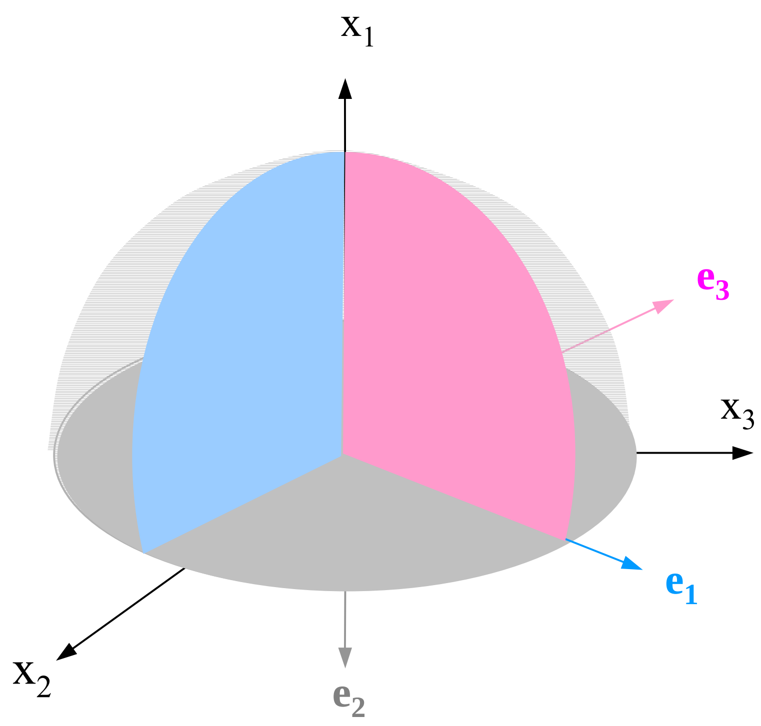
Results of Reconstructed Elastic Constants for aluminum oxide (II)

- The average of the eigenvectors of A and B is used as good estimates of the normals to the symmetry planes.

$$V^{ave} = \begin{bmatrix} -0.0087 & -1.0000 & -0.0158 \\ 0.2405 & 0.0134 & -0.9701 \\ 0.9702 & -0.0115 & 0.2418 \end{bmatrix}$$

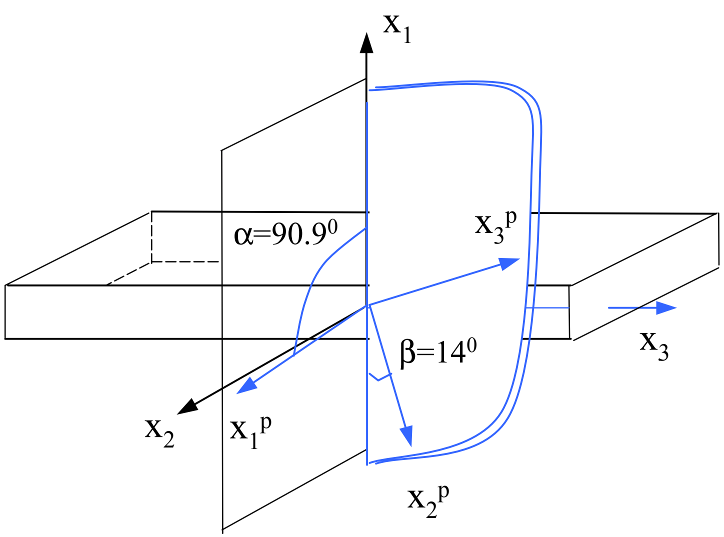
- Therefore, the unit vectors of the normals to symmetry planes are

$$e_1 = -0.0087 x_1 + 0.2405 x_2 + 0.9702 x_3$$
$$e_2 = -1.0000 x_1 + 0.0134 x_2 - 0.0115 x_3$$
$$e_3 = -0.0158 x_1 - 0.9701 x_2 + 0.2404 x_3$$



Results of Estimation of Principal Coordinate System R^P

- R^P can be located by determining a set of Euler's angles with at least one of the angles is zero.
- V_3^{ave} was used to extract the Euler's angles.



$$e_3 = -0.0158x_1 - 0.9701x_2 + 0.2404x_3$$
$$= \cos(\alpha)\,x_1 - \sin(\alpha)\cos(\beta)\,x_2 + \sin(\alpha)\sin(\beta)\,x_2$$

Thus

$$\alpha \cong 90.9^0, \beta \cong 14.0^0, \gamma = 0$$

Results of Estimation of Principal Coordinate System R^P (II)

- Elastic constants of aluminum oxide in R^P are (sample No.1):

$$
\left[C^p\right]=\begin{bmatrix}
478.5(495) & 178.9(160) & 121.9(115) & -1.882(-23) & -23.54(0) & 0.801(0) \\
 & 471.9(495) & 114.9(115) & 2.063(23) & 35.02(0) & 0.910(0) \\
 & & 354.5(497) & 0.210(0) & -13.33(0) & 0.110(0) \\
 & & & 9.170(146) & 2.139(0) & 47.87(0) \\
 & sym. & & & 144.5(146) & -1.681(-23) \\
 & & & & & 175.3(168)
\end{bmatrix}
$$

The numbers in the parenthesis in Equation 7.9 are the elastic constants obtained from ref.

- Reconstructed values C_{11}, C_{13}, C_{16}, C_{22}, C_{23}, C_{26}, C_{34}, C_{36}, C_{45}, C_{55}, C_{66} are within ± 5 % of handbook values. C_{12} is within 11.5 %.

Results with the Carbon-Carbon Composites

- Three carbon-carbon samples were tested, initial guess (in GPa) for the optimization algorithm is:

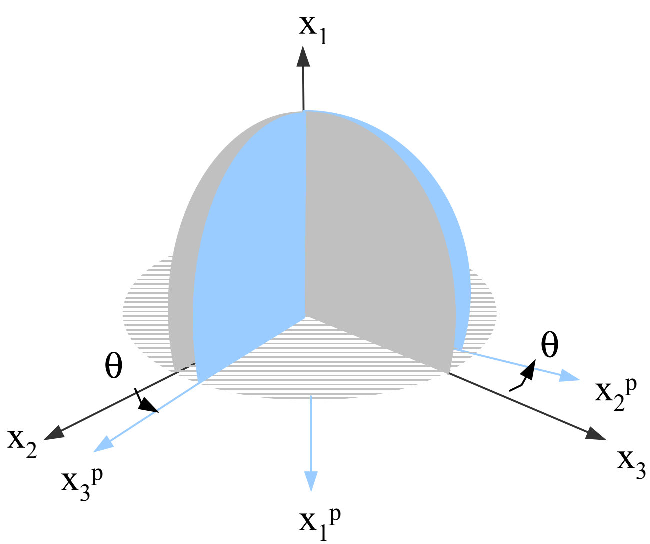
$$[C]=\begin{bmatrix} 47.601 & 44.371 & 46.342 & 0 & 0 & 0 \\ & 18.906 & 20.327 & 0 & 0 & 0 \\ & & 18.839 & 0 & 0 & 0 \\ & & & 3.1112 & 0 & 0 \\ & \text{sym.} & & & 5.2180 & 0 \\ & & & & & 4.5369 \end{bmatrix}$$

- Reconstructed elastic constants in R are (sample No. 1):

$$[C]=\begin{bmatrix} 47.440 & 44.296 & 46.160 & -0.3010 & 0.0081 & 0.0010 \\ & 19.113 & 20.000 & 0.0111 & 0.0000 & 0.0300 \\ & & 18.968 & -0.001 & 0.0283 & -0.0010 \\ & & & 5.6274 & 0.2940 & 0.0000 \\ & \textit{sym.} & & & 6.0371 & -0.2111 \\ & & & & & 4.5809 \end{bmatrix}$$

- Euler's angles corresponding to R with respect to R^P are

$$\alpha \cong 179.6^0, \beta \cong 80.7^0, \gamma = 0$$



Misorientation between geometric axis and symmetric axis $\theta = 9.3^0$.

Results with the Carbon-Carbon Composites (II)

Results of elastic constants (GPa) for sample No.2 and No. 3 are:

$$[C]=\begin{bmatrix} 76.54 & -27.18 & 27.99 & 0.010 & 0.000 & 0.010 \\ & -28.49 & -28.26 & 0.021 & 0.000 & 0.003 \\ & & 25.33 & -0.000 & 2.503 & -0.001 \\ & & & 17.10 & 5.830 & 0.000 \\ & sym. & & & 8.615 & 0.002 \\ & & & & & -1.658 \end{bmatrix}$$

$$[C]=\begin{bmatrix} 1.994 & -25.93 & 5.382 & -0.000 & -0.000 & -0.010 \\ & 9.109 & -32.11 & 0.000 & -0.004 & 0.001 \\ & & 34.38 & -0.003 & -2.043 & 0.000 \\ & & & 17.76 & 5.755 & -0.004 \\ & sym. & & & 8.968 & -0.002 \\ & & & & & 5.237 \end{bmatrix}$$

		initial data	sample 1	sample 2	sample 3
Euler's angles	α	N/A	179.6^0	6.3^0	93.5^0
	β	N/A	80.7	90.3^0	89.7^0
numbers of symmetry planes		3 perpendicular symmetry planes	3 perpendicular symmetry planes	3 perpendicular symmetry planes	1symmetry plane
deviation of geometric axes from materials sym. axes		0^0	9.3^0	6.3^0	6.5^0

Conclusions

- Water immersion method to optimally recover 21 elastic constants for a general anisotropic material has been demonstrated.
- Approach is based on wave velocity measurements and a Newton-Raphson nonlinear optimization.
- Identification of martial symmetries and the corresponding principal coordinate system has been introduced.
- Numerical as well as experimental results show the method introduced in this work is effective for determining misorientation of geometric axes from its symmetric axes in carbon-carbon composite.

Future Work

- Study the stability of optimization algorithm. Apply random scatter on the initial guess and evaluate error in the measured velocity data.
- Investigate the sensitivity of this method for the case of weak and strong anisotropies (quasi-isotropic E-glass vinylester thick composites and wood)
- For specimens ranging from wood to engineering composites, many interesting results are expected.
- Future work also consider extending the determination of symmetry class to non-Cartesian samples.
  - *This is an interesting topic and one with relevance to a range of nature and man-made materials.*

Acknowledgement

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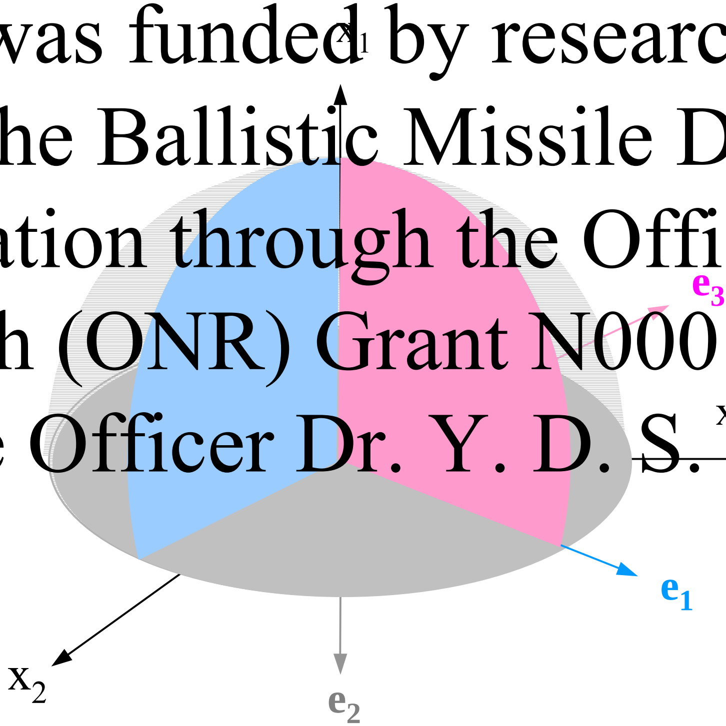
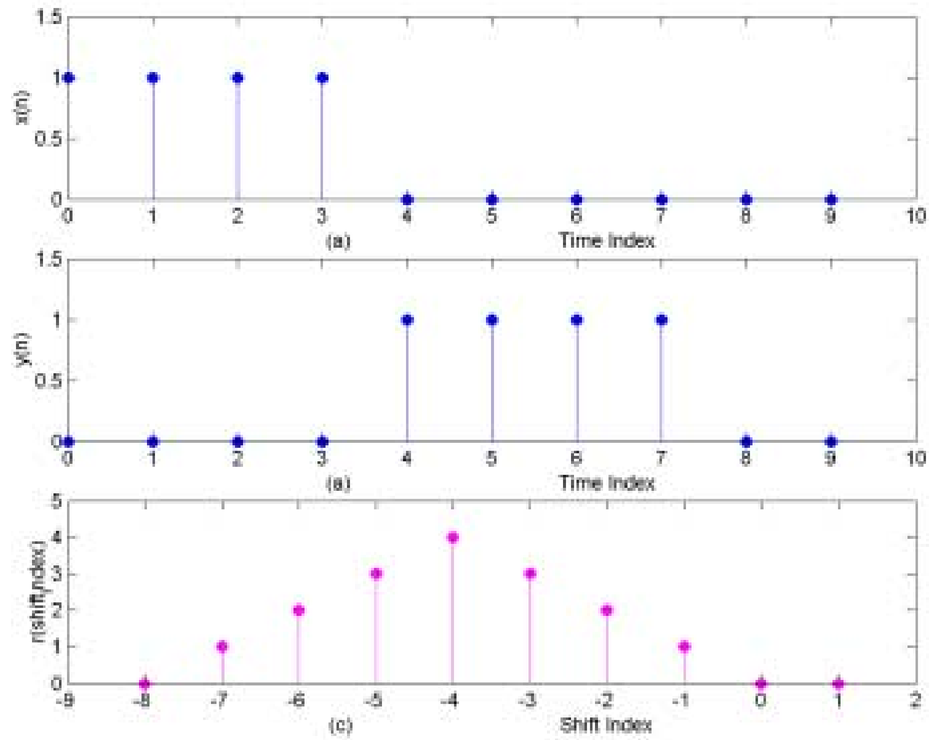


Cross Correlation—Relative Time Delay Estimation

$$r_{xy}(\tau) = \sum_{n=-\infty}^{+\infty} x(n)y(n-\tau)$$

- Cross correlation is a mathematical operation that is to measure the similarity of two functions.
- The maximum of the cross correlation indicates that the functions being correlated are most similar.



Cross Correlation (II)

- Cross correlation is used to measure the relative time delay between a reference signal and an unknown signal.
- The relative time delay is determined by locating the peak of the cross correlation function.
- The phase velocity is estimated:

$$V_i(x_1,\varphi) = \frac{V_0}{\sqrt{1+\frac{V_0\tau_i}{d}\left[\frac{V_0\tau_i}{d} - 2\cos\theta_i\right]}}$$