Thermally Induced Stresses in Functionally Graded Thick Tubes

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Abstract. An analytical solution is developed for isotropic functionally graded tubes that are subjected to steady thermomechanical loads. The heat conduction and thermoelasticity equations are solved analytically using the power series method to obtain the temperature, displacements and thermal stresses. In addition to the thermal and mechanical boundary conditions at the inner and outer surfaces of the tube, the net axial force acting on the tube is set equal to zero. The solution technique is applicable to functionally graded tubes with arbitrary variation of material properties in the radial direction. Results are presented for a two-constituent isotropic metal/ceramic functionally graded tube that is subjected to a very high temperature on its inner surface.

Keywords: Thick FGM tube, heat conduction, thermoelasticity, analytical solution.

INTRODUCTION

Functionally graded materials (FGMs) are advanced composite materials that are engineered to have a smooth spatial variation of material properties. This is achieved by fabricating the composite material to have a gradual spatial variation of the constituent materials' relative volume fractions and microstructure, thus tailoring its material composition based on functional performance requirements [1]. FGMs offer great promise in applications where the operating conditions are severe, including spacecraft heat shields, heat exchanger tubes, plasma facings for fusion reactors and engine components. For example, in a conventional thermal barrier coating for high temperature applications, a discrete layer of ceramic material is bonded to a metallic structure. However, the abrupt transition in material properties across the interface between distinct materials can cause large interlaminar stresses and lead to plastic deformation or cracking [2]. These adverse effects can be alleviated by functionally grading the material to have a smooth spatial variation of material composition, with ceramic-rich material placed at the high temperature locations and metal-rich material placed at regions where mechanical properties, such as toughness, need to be high.

Several analytical solutions have been developed for the analysis of FGM tubes subjected to thermomechanical loads. However, to the best of our knowledge, the available analytical solution techniques are not applicable to FGM tubes with arbitrary radial variation of material properties. Liew et al. [3] have used a layer-wise analysis where each layer is assumed to be homogeneous. Zimmerman and Lutz [4] have solved the plane strain problem of an isotropic solid cylinder under uniform heating using the Frobenius method. The formulation uses the Lame parameters; the elastic

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moduli and the thermal expansion coefficient are assumed to vary linearly with radius. The two-dimensional steady state thermal stresses in a hollow, thick functionally graded cylinder have been analyzed by Jabbari et al. [5]. The material properties have a power law dependence on the radial coordinate; the Poisson's ratio however is considered to be constant. Several researchers (e.g. see [6-7]) have used the rule of mixtures based on the volume fractions of the constituents, without taking recourse to micromechanics models for the homogenization of material properties. It is important to note that when manufacturing FGMs it is possible to control only the volume fraction profile and microstructure of the material phases. The material properties (thermal conductivity, thermal expansion coefficient, Young's modulus and Poisson's ratio) at a point would need to be inferred from the local volume fractions and morphology of the constituent phases using rigorous micromechanical homogenization theories, such as the self-consistent scheme [8]. In general, the resulting Young's modulus and Poisson's ratio are complicated functions of the radial coordinate.

The objective of the present investigation is to provide an analytical solution for functionally graded tubes with arbitrary variation of material properties in the radial direction. The effective material properties are determined from the local volume fractions and material properties of the isotropic phases using the self-consistent scheme [8]. The solution technique is equally applicable to other homogenization schemes. Results are presented for isotropic titanium/zirconia FGM tube that is exposed to a high temperature on its inner surface.

PROBLEM FORMULATION

The geometry of a functionally graded cylindrical shell is described using a global, cylindrical x-θ-r coordinate system, with coordinates x, θ and r denoting the axial, circumferential and radial coordinate directions, respectively. The tube is made of an isotropic material with material properties varying smoothly in the radial direction only. The temperature and displacement fields are assumed to be axisymmetric. Furthermore, the functionally graded tube is assumed to be of infinite extent in the axial direction and the heat flux and stresses are taken to be independent of x. The analysis is based on the quasi-static linear theory of thermoelasticity. The steady state heat equation in the absence of internal heat sources is

$$\frac{dq_r}{dr} + \frac{q_r}{r} = 0, \tag{1}$$

and the mechanical equilibrium equations in the absence of body forces reduce to

$$\begin{aligned} &\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \\ &\frac{d\sigma_{r\theta}}{dr} + \frac{2}{r}\sigma_{r\theta} = 0, \\ &\frac{d\sigma_{rx}}{dr} + \frac{1}{r}\sigma_{rx} = 0. \end{aligned} \tag{2}$$

Here q and σ denote the components of the heat flux vector and stress tensor, respectively. Constitutive equations for a linear thermoelastic material are

$$q_r = -\kappa\, T'(r), \tag{3}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{rr} \\ \sigma_{\theta r} \\ \sigma_{rx} \\ \sigma_{x\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \alpha T \\ \varepsilon_{\theta\theta} - \alpha T \\ \varepsilon_{rr} - \alpha T \\ 2\varepsilon_{\theta r} \\ 2\varepsilon_{rx} \\ 2\varepsilon_{x\theta} \end{Bmatrix}, \tag{4}$$

where

$$C_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)},\ C_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)},\ C_{44} = \frac{E}{2(1+\nu)}, \tag{5}$$

T is the change in temperature from the stress-free reference temperature, κ is the thermal conductivity, E is the Young's modulus, ν is the Poisson's ratio, ε denotes the components of the infinitesimal strain tensor and α is the coefficient of thermal expansion.

The following thermal and mechanical boundary conditions are prescribed on the inner and outer surfaces of the tube

$$\begin{aligned} T = T_i,\ \sigma_{rr} = \sigma_{\theta r} = \sigma_{rx} = 0 \ \text{ at } \ r = R_i, \\ T = T_o,\ \sigma_{rr} = \sigma_{\theta r} = \sigma_{rx} = 0 \ \text{ at } \ r = R_o, \end{aligned} \tag{6}$$

where R_i and R_o are the inner surface and outer surface radii, respectively. In addition, the net axial force on the tube cross section is zero,

$$2\pi \int_{R_i}^{R_o} \sigma_{xx}(r) r dr = 0. \tag{7}$$

AN ANALYTICAL SOLUTION

It is assumed that the material properties are analytic functions of r and thus can be represented by a Taylor series expansion about the midsurface as,

$$[\kappa(r), C_{ij}(r), \alpha(r)] = \sum_{a=0}^{\infty} [\kappa^{(a)}, C_{ij}^{(a)}, \alpha^{(a)}]\,(r-R)^a, \tag{8}$$

where $R = (R_i+R_o)/2$.

A power series solution for the change in temperature is sought in the form

$$T(r) = \sum_{b=0}^{\infty} \tilde{T}^{(b)} (r-R)^b. \tag{9}$$

Substitution for T from (9) into (3) and the result into (1), yields a recurrence relation to obtain the coefficients $\tilde{T}^{(b)}$ ($b \geq 2$) in terms of $\tilde{T}^{(0)}$ and $\tilde{T}^{(1)}$. The two constants $\tilde{T}^{(0)}$ and $\tilde{T}^{(1)}$ are obtained by satisfying the temperature boundary conditions (6) on the inner and outer surfaces of the hollow cylinder.

Solving the differential equations $(2)_2$ and $(2)_3$ for $\sigma_{r\theta}$ and σ_{rx} and enforcing the traction-free boundary conditions on the inner and outer surfaces, reveals that the shear stress components $\sigma_{r\theta}$ and σ_{rx} are identically zero throughout the functionally graded tube. Next, the strains, which are obtained from the stresses, are integrated to obtain the axial displacement u,

$$u(x,r) = Bx + P, \tag{10}$$

where B and P are constants that represent the axial strain and the rigid body translation, respectively. A power-series solution to the radial displacement is sought by assuming the following form for $w(r)$

$$w(r) = \sum_{b=0}^{\infty} w^{(b)} (r - R)^b. \tag{11}$$

Substitution of (11) into the radial equilibrium equation $(2)_1$ yields a recurrence relation for the coefficients $w^{(b)}$ ($b \geq 2$) in terms of $w^{(0)}$ and $w^{(1)}$. The constants $w^{(0)}$ and $w^{(1)}$ are determined from the boundary conditions (6) for σ_{rr} on the inner and outer surfaces of the functionally graded tube.

RESULTS AND DISCUSSION

Representative results are presented for a titanium/zirconia FGM tube. The tube is exposed to 1200 K on its inner surface while the outer surface of the tube is maintained at the stress free reference temperature of 300 K. The high-temperature inner surface of the tube is composed of partially stabilized zirconia (PSZ) and the outer surface is made of Ti-6Al-4V alloy. The volume fraction of PSZ is assumed to vary linearly in the radial direction

$$V_{PSZ} = (R_o - r)/(R_o - R_i). \tag{12}$$

The inner and outer radii are assumed to be R_i=0.05 m and R_o=0.1 m, respectively. The thermal conductivity, thermal expansion coefficient, Young's modulus and Poisson's ratio of Ti-6Al-4V and PSZ at the reference temperature of 300 K are given in Table 1 [9]. The homogenized material properties at a point are computed using the self-consistent scheme wherein the effective bulk and shear moduli are obtained by solving a quartic equation [8, 10-11]. It is worth noting that although the assumed volume fraction profile (12) has a linear variation in the radial direction, the homogenized Young's modulus and Poisson's ratio are complicated nonlinear functions of the radial coordinate [12].

The yield strength of Ti-6Al-4V and the ultimate tensile and compressive strength of PSZ are also listed in Table 1 [13, 14].

TABLE 1. Titanium and Zirconia Material Properties.

Material	κ (W/mK)	α (10^{-6}/K)	E (GPa)	ν	S_y (MPa)	S_{ut} (MPa)	S_{uc} (MPa)
Ti-6Al-4V	6.200	8.856	105.75	0.298	816	-	-
PSZ	1.783	8.783	116.38	0.333	-	352	1750

The variations of the temperature and heat flux in the radial direction are shown in Fig. 1. The temperature at the midsurface is 542.8 K. The radial displacement, shown in Fig. 2(a), reaches a maximum value near the mid-surface of the tube. The hoop, axial and radial stresses are shown in Fig. 2(b)-(d). The hoop and axial stresses exhibit similar trends and they are compressive on the inner, ceramic surface of the tube.

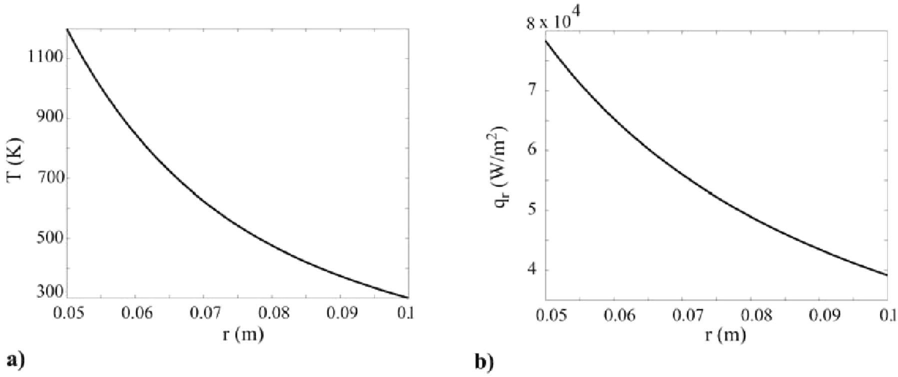

FIGURE 1. Radial variation of temperature and radial heat flux component.

The hoop and axial stresses are -957.75 MPa and -960.79 MPa, respectively, on the inner surface of the tube. Using the modified-Mohr theory of brittle failure [15], the factor of safety at the inner surface of the tube is 1.82. The magnitude of the hoop and axial stresses are 357.88 MPa and 359.49 MPa, respectively, on the outer surface of the tube. Based on the von Mises-Hencky distortion-energy failure theory for ductile materials, the factor of safety at the titanium outer surface is 2.28.

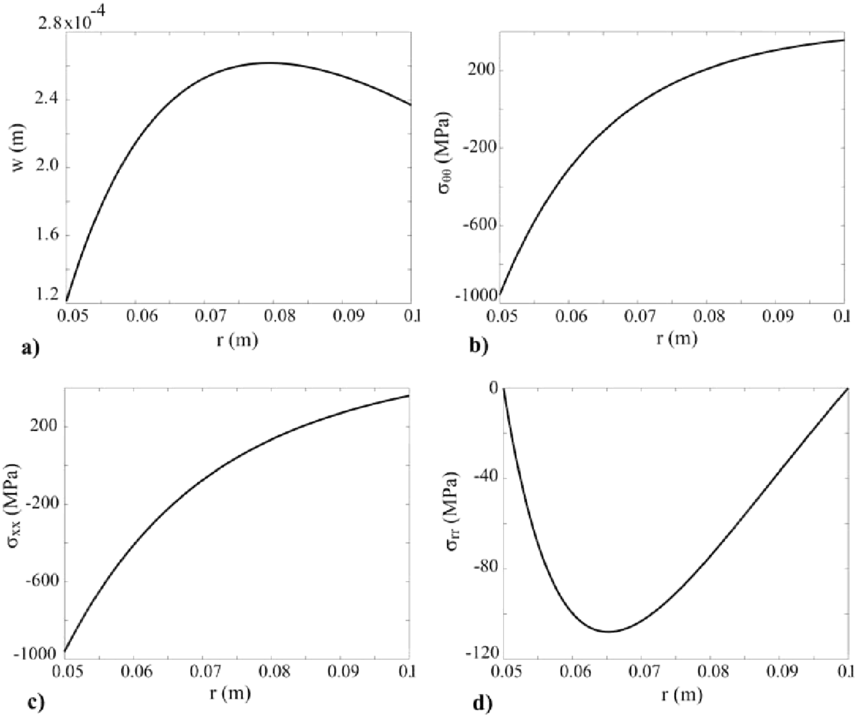

FIGURE 2. Radial variation of (a) radial displacement, (b) hoop stress, (c) axial stress and (d) normal stress in the radial direction.

CONCLUSIONS

An analytical solution has been obtained for the steady-state temperature and stresses in FGM tubes for arbitrary variation of material properties in the radial direction. The differential equations of heat conduction and thermoelasticity are solved using a power series solution technique. Results are presented for a thick metal/ceramic FGM tube that is exposed to a high temperature on its inner surface. The generality of the solution technique for arbitrary variations of material properties in the radial direction will enable the designer to tailor the volume fraction profile to increase the strength-to-weight ratio of functionally graded tubes.

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